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Soliton gauge states and T-duality of closed bosonic string compatified on torus

J.-C. Lee[a]

Department of Physics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139, USA (e-mail: jcclee@cc.netu.edu.tw)

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Abstract. We study soliton gauge states in the spectrum of bosonic string compatified on torus. The enhenced Kac-Moody gauge symmetry, and thus T-duality, is shown to be related to the existence of these soliton gauge states in some moduli points.

1 Introduction

String duality [1] has been the subject of active research for the last few years. The five consistent perturbative string theories are now known to be related to each other through various duality symmetries. It is believed that they are merely different moduli points of a single underlying theory termed M-theory. The best known string duality is the T-duality which can be understood perturbatively [2]. T-duality relates a string theory in a background with large volume to another string theory in a background with small volume. For example, it has been shown that the Heterotic $E_8 \otimes E_8$ and SO(32) theories sit at different points, which are T-dual to each other, of the moduli space of the same Heterotic theory below ten dimension [3].

For the compatified bosonic string, the discrete T-duality group were shown to be the residual Weyl subgroup of the enhenced Kac-Moody gauge symmetry [2]. On the other hand, it has been known that space-time gauge symmetry of uncompatified string is related to the existence of gauge states in the spectrum [4]. For the 10D Heterotic string, the Heterotic gauge states [5] are responsible for the massless $E_8 \otimes E_8$ or SO(32) gauge symmetry and are used to predict the existence of an infinite number of massive Einstein-Yang-Mills type gauge symmetry. For the toy 2D string, the discrete gauge states [6] are responsible for the w_∞ symmetry of the Liouville theory. It is thus of interest to understand the gauge state structure of the compatified string theory, and study their relation to the enhenced Kac-Moody gauge symmetry.

In this paper, for simplicity, we will study gauge states of closed bosonic string compatified on torus. In addition to the usual gauge states, we will discover soliton gauge states (SGS) in the spectrum of some moduli points. These gauge states and SGS form a realization of enhenced Kac-Moody gauge symmetry group in the gauge state sector of the spectrum. Since T-duality group is the Weyl subgroup of the enhenced gauge group, SGS can be considered as the origin of the discrete T-duality group. In Sect. 2, we derive massless gauge states of bosonic string compatified on $R^{25} \otimes T^1$ at self-dual point $R = \sqrt{2}$, and show that they form a representation of the enhenced $SU(2)_R \otimes SU(2)_L$ gauge group. In Sect. 3, we generalize the calculation to $R^{26-D} \otimes T^D$ and give examples at some moduli points. Section 4 is devoted to the discussion of massive SGS. We will find that there is an infinite number of massive SGS which exists at some moduli points. The existence of these massive SCS implies that there is an infinite enhenced gauge symmetry of compatified string theory. Finally a brief conclusion is given in Sect. 5.

2 Soliton gauge state on $R^{25} \otimes T^1$

In the simplest torus compatification, one coordinate of the string was compatified on a circle of radius R [7]

$$X^{25}(\sigma + 2\pi, \pi) = X^{25}(\sigma, \pi) + 2\pi R n \,. \qquad (2.1)$$

Singlevaluedness of the wave function then restricts the allowed momenta to be $p^{25} = m/R$ with $m, n \in Z$. The mode expansion of the compatified coordinate for right (left) mover is

$$X_R^{25} = \frac{1}{2}x^{25} + \left(p^{25} - \frac{1}{2}nR\right)(r-\sigma) + i\sum_{r\neq 0}\frac{1}{r}\alpha_r^{25}e^{-ir(r-\sigma)} \,, \qquad (2.2)$$

$$X_L^{25} = \frac{1}{2}x^{25} + \left(p^{25} + \frac{1}{2}nR\right)(r-\sigma) + i\sum_{r\neq 0}\frac{1}{r}\alpha_r^{25}e^{-ir(r+\sigma)} \,. \qquad (2.3)$$

We have normalized the string tension to be $\frac{1}{4\pi T} = 1$ or $\alpha' = 2$. The Virasoro operators can be written as

$$L_0 = \frac{1}{2}\left(p^{25} - \frac{1}{2}nR\right) + \frac{1}{2}p^{\mu^2} + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \,, \qquad (2.4)$$

[a] On leave of absence from: Department of Electrophysics, National Chiao-Tung University, Taiwan, R.O.C.

$$\tilde{L}_0 = \frac{1}{2}\left(p^{25} + \frac{1}{2}nR\right) + \frac{1}{2}p^{\mu^2} + \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n \ , \quad (2.5)$$

and

$$L_m = \frac{1}{2}\alpha_0^2 + \sum_{-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \ , \quad (2.6)$$

$$\tilde{L}_m = \frac{1}{2}\tilde{\alpha}_0^2 + \sum_{-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \quad (m \neq 0) \quad (2.7)$$

where

$$\alpha_0^{25} = p^{25} - \frac{1}{2}nR \equiv p_R^{25} \ , \quad (2.8)$$

$$\tilde{\alpha}_0^{25} = p^{25} + \frac{1}{2}nR \equiv p_L^{25} \ , \quad (2.9)$$

and the 25d momentum is $\alpha_0^\mu = \tilde{\alpha}_0^\mu = p^\mu \equiv k^\mu$. In the old covariant quantization of the theory, in addition to the physical propagating states, there are four types of gauge states in the spectrum

$$\text{I.a } |\psi\rangle = L_{-1}|\chi\rangle \quad \text{where} \quad L_m\chi\rangle = 0 \ , \ (\tilde{L}_m - \delta_m)|\chi\rangle = 0 \ , \quad (m = 0, 1, 2, \ldots) \quad (2.10)$$

$$\text{II.a } |\psi\rangle = \left(L_{-2} + \frac{3}{2}L_{-1}^2\right)|\chi\rangle \text{ where } (L_m + \delta_m)|\chi\rangle = 0 \ , \quad (\tilde{L}_m - \delta_m)|\chi\rangle = 0; , \quad (m = 0, 1, 2, \ldots) \quad (2.11)$$

and by interchanging all left and right mover operators, one gets I.b and II.b states. Type II states are zero-norm gauge states only at critical space-time dimension. We will only calculate type a states. Similar results can be easily obtained for type b states. For type I.a state, the $m = 0$ constraint of (2.10) gives

$$M^2 = \frac{m^2}{R^2} + \frac{1}{4}n^2R^2 + N + \tilde{N} - 1 \ , \quad (2.12)$$

$$N - \tilde{N} = mn - 1 \quad (2.13)$$

where $N \equiv \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n$ and $\tilde{N} \equiv \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$. For massless $M^2 = 0$ states, $N + \tilde{N} = 0$ or 1. The solutions of (2.12) and (2.13) are

$$N = 0 \ , \ \tilde{N} = 1 \ , \ m = n = 0 \quad (\text{any } R) \quad (2.14)$$

$$N = \tilde{N} = 0 \ , \ m = n = \pm 1 \quad R = \sqrt{2} \ . \quad (2.15)$$

Equation (2.15) gives us our first SGS. It is easy to write down the explicit form of $|\chi\rangle$ and $|\psi\rangle$, and impose the $m \neq 0$ constraints of (2.10). There are also a vector and a scalar gauge states in (2.14). Similar results can be obtained for the type I.b state. In this case, $m = -n = \pm 1$. There is no type II solution in the massless case. We note that there are massless soliton gauge states only when $R = \sqrt{2}$ which is known as self-dual point in the moduli space. The vertex operators of all gauge states are calculated to be

$$k_\mu\theta_\nu\partial X_R^\mu\bar{\partial}X_L^\nu e^{ikx}; \quad L \leftrightarrow R, \quad (2.16)$$

$$k_\mu\partial X_R^\mu\bar{\partial}X_L^{25}e^{ikx}, \quad (2.17)$$

$$k_\mu\bar{\partial}X_L^\mu\partial X_R^{25}e^{ikx}, \quad (2.18)$$

$$k_\mu\partial X_R^\mu e^{\pm i\sqrt{2}X_L^{25}}e^{ikx}, \quad (2.19)$$

$$k_\mu\bar{\partial}X_L^\mu e^{\pm i\sqrt{2}X_R^{25}}e^{ikx}. \quad (2.20)$$

It is easy to see that the three gauge states of (2.18) and (2.20) form a representation of $SU(2)_R$ Kac-Moody algebra. Similarly, (2.17) and (2.19) form a representation of $SU(2)_L$ Kac-Moody algebra. The vector gauge states in (2.16) are responsible for the gauge symmetry of graviton and antisymmetric tensor field. We see that the self-dual point $R = \sqrt{2}$ is very special even from the gauge sector point of view.

3 Soliton gauge state on $R^{26-D} \otimes T^D$

In this section we compatify D coordinates on a D-dimensional torus $T^D \equiv \frac{R^D}{2\pi\Lambda^D}$

$$\vec{X}(\sigma + 2\pi, \pi) = \vec{X}(\sigma, \pi) + 2\pi\vec{L}. \quad (3.1)$$

with

$$\vec{L} = \sum_{i=1}^{D} n_i \left(R_i\frac{\vec{e}_i}{\sqrt{2}}\right) \in \Lambda^D \quad (3.2)$$

where Λ^D is a D-dimensional lattice with a basis $\left\{R_1\frac{\vec{e}_1}{\sqrt{2}}, R_2\frac{\vec{e}_2}{\sqrt{2}}, \ \ldots, R_D\frac{\vec{e}_D}{\sqrt{2}}\right\}$. We have chosen $|\vec{e}_i|^2$. The allowed momenta $\vec{p}$ take values on the dual lattice of Λ^D

$$\vec{p} = \sum_{i=1}^{D} m_i \left(\frac{1}{R_i}\sqrt{2}\vec{e}_i^{\,*}\right) \in (\Lambda^D)^*. \quad (3.3)$$

The basis of $(\Lambda^D)^*$ is $\left\{\frac{1}{R_1}\sqrt{2}\vec{e}_1^{\,*}, \ \frac{1}{R_2}\sqrt{2}\vec{e}_2^{\,*}, \ \ldots, \frac{1}{R_D}\sqrt{2}\vec{e}_D^{\,*}, \right\}$ and we have $\vec{e}_i\cdot\vec{e}_j^{\,*} = \delta_{ij}$. The mode expansion of the compatified coordinates is

$$\vec{X}_R = \frac{1}{2}\vec{x} + \left(\vec{p} - \frac{1}{2}\vec{L}\right)(\tau - \sigma) + i\sum_{r\neq 0}\frac{1}{r}\alpha_r^{25}e^{-ir(\tau-\sigma)} \ , \quad (3.4)$$

$$\vec{X}_L = \frac{1}{2}\vec{x} + \left(\vec{p} - \frac{1}{2}\vec{L}\right)(\tau + \sigma) + i\sum_{r\neq 0}\frac{1}{r}\alpha_r^{25}e^{-ir(\tau+\sigma)} \ , \quad (3.5)$$

The right and left momenta are defined to be $\vec{p}_R = \left(\vec{p} - \frac{1}{2}\vec{L}\right)$ and $\vec{p}_L = \left(\vec{p} + \frac{1}{2}\vec{L}\right)$ It can be shown that the 2D-vector $(\vec{p}_R, \vec{p}_L)$ build an even self-dual Lorentzian lattice $\Gamma_{D,D}$, which guarantees the string one loop modular invariance of the theory [8]. The moduli space of the theory is

$$\mu = \frac{SO(D,D)}{SO(D)\times SO(D)}/O(D,D,Z) \quad (3.6)$$

where $O(D,D,Z)$ is the discrete T-duality group and dim $\mu = D^2$. To complete the parametrization of the moduli space, one needs to introduce an antisymmetric tensor field B_{ij} in the bosonic string action. This will modify the right (left) momenta to be

$$\vec{p}_R = \left(\vec{p}_B - \frac{1}{2}\vec{L}\right), \tag{3.7}$$

$$\vec{p}_L = \left(\vec{p}_B + \frac{1}{2}\vec{L}\right) \tag{3.8}$$

where

$$\vec{p}_B = \sum_{ij}\left(m_i \frac{1}{R_i}\sqrt{2}\vec{e}_i^* - n_j \frac{1}{\sqrt{2}R_i} B_{ij}\vec{e}_i^*\right). \tag{3.9}$$

We are now ready to discuss the gauge state. As a first step, we restrict ourselves to moduli space with $B_{ij} = 0$. For the type I.a state, the $m = 0$ constraint of (2.10) for massless states gives

$$N + \tilde{N} + \vec{p}^2 + \frac{1}{4}\vec{L}^2 = 1, \tag{3.10}$$

$$N - \tilde{N} = \sum_i m_i n_i - 1. \tag{3.11}$$

It is easy to see $N + \tilde{N} = 0$ or 1. For $N + \tilde{N} = 1, m_i = n_i = 0$, we have trivial gauge state solutions. SGS exists for the case $N = \tilde{N} = 0$ and the following moduli points

$$R_i = \sqrt{2}, e_i^I = \sqrt{2}\delta_i^I \quad (i = 1, 2, \ldots, d) \tag{3.12}$$

with $m_i = n_i = \pm 1$, and $m_j = n_j = 0$ for $d < j \leq D$. In each case, the gauge states and SGS form a representation of $SU(2)^d$ algebra. Similar results can be easily obtained for the type I.b SGS. As in Sect. 2, there is no massless type II SGS. We now discuss $B_{ij} \neq 0$ case. For illustration, we choose $D = 2$. In this case $B_{ij} = B\epsilon_{ij}$, and one has four moduli parameters $R_, R_2, B$, and $\vec{e}_1 \cdot \vec{e}_2$. For type I.a state, the $m = 0$ constraint of (2.10) gives

$$N + \tilde{N} + \vec{p}_B^2 + \frac{1}{4}\vec{L}^2 = 1, \tag{3.13}$$

$$N - \tilde{N} = m_1 n_1 + m_2 n_2 - 1. \tag{3.14}$$

SGS exists only for $N = \tilde{N} = 0$. For the moduli point

$$R_1 = R_2 = \sqrt{2},\ B = \frac{1}{2},\ \vec{e}_1 = (\sqrt{2}, 0),$$
$$\vec{e}_2 = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}\right), \tag{3.15}$$

one gets six SGS with momenta $\vec{p}_R$ being the six root vectors of $SU(3)_R$. Together with two other trivial gauge states corresponding to $N = 0$, $\tilde{N} = 1$, they form the Frenkel-Kac-Segal [9] representation of $SU(3)_{k=1}$ Kac-Moody algebra. Note that $\vec{e}_1, \vec{e}_2$ are the two simple roots of SU(3) and $\vec{e}_1^* = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{6}}\right)$, $\vec{e}_2^* = \left(0, \sqrt{\frac{2}{3}}\right)$. The six sets of winding number are $(m_1, n_1, m_2, n_2) = (1, 1, 0, 0)$, $(-1, -1, 0, 0)$, $(0, 0, 1, 1)$, $(0, 0, -1, -1)$, $(1, 1, 1, 0)$, $(-1, -1, -1, 0)$. Similar results can be obtained for type I.b SGS. The gauge states (including SGS) thus form a representation of enhenced $SU(3)_R \otimes SU(3)_L$ at the moduli point of (3.15). In general, we expect that all enhenced Kac-Moody gauge symmetry at any moduli point should have a realization on SGS.

4 Massive soliton gauge state

In this section we derive the massive SGS at the first massive level $M^2 = 2$. We will find that SGS exists at infinite number of moduli points. One can also show that they exist at an infinite number of massive level. The existence of these massive SGS implies that there is an infinite enhenced gauge symmetry structure of compatified string theory. For type I.a state, the $m = 0$ constraint of (2.10) gives

$$\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 + N + \tilde{N} = 3, \tag{4.1}$$

$$N - \tilde{N} = mn - 1. \tag{4.2}$$

which implies $N + \tilde{N} = 0, 1, 2, 3$. Equation (4.1) and (4.2) can be easily solved as following:

$$1.\ N + \tilde{N} = 3 : m = n = 0, N = 1, \tilde{N} = 2, \quad \text{any } R. \tag{4.3}$$

$$2.\ N + \tilde{N} = 2 : mn = 1, N = \tilde{N} = 1, \quad R = \sqrt{2}, \tag{4.4}$$
$$mn = -1, N = 0, \tilde{N} = 2, \quad R = \sqrt{2}.$$

$$3.\ N + \tilde{N} = 1 : mn = 2, N = 1, \tilde{N} = 0,$$
$$R = 2, 1. (\text{ T-duality})$$
$$mn = 0, N = 0, \tilde{N} = 1, \tag{4.5}$$
$$R = \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}. (\text{T-duality})$$

$$4.\ N + \tilde{N} = 0 : mn = 1, N = \tilde{N} = 1,$$
$$R = 2 \pm \sqrt{2}(\mathrm{T-duality}) \tag{4.6}$$

where we have included a T-duality transformation $R \to \frac{2}{R}$ for some moduli points. Note that (4.5) tells us that massive SGS exists at an infinite number of moduli point. For type II.a state, the $m = 0$ constraint of (2.11) gives

$$\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 + N + \tilde{N} = 2, \tag{4.7}$$

$$N - \tilde{N} = mn - 2, \tag{4.8}$$

which implies $N + \tilde{N} = 0, 1, 2$. Equation (4.7) and (4.8) can be solved as following:

$$1.\ N + \tilde{N} = 2 : m = n = 0, N = 0, \tilde{N} = 2, \quad \text{any } R. \tag{4.9}$$

$$2.\ N + \tilde{N} = 1 : mn = 1, N = 0, \tilde{N} = 1, \quad R = \sqrt{2} : \tag{4.10}$$

$$3.\ N + \tilde{N} = 0 : mn = 2, N = \tilde{N} = 0,$$
$$R = 2, 1. (\text{T-duality}) \tag{4.11}$$

The vertex operators of all SGS can be easily calculated and written down. Similar results can be obtained for type b gauge state. One can also calculate propagating soliton states by using the same technique. We summarize the moduli points which exist soliton state and SGS as following:

$$a. \text{Soliton gauge state} : R = \sqrt{2}, 2 \pm \sqrt{2}, \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}, 2, 1. \tag{4.12}$$

$$b.\,\text{Soliton state}\,:\; R=\sqrt{2}, 2\pm\sqrt{2}, \frac{|m|}{\sqrt{2}}, \frac{2\sqrt{2}}{|m|}, \frac{|m|}{2}, \frac{4}{|m|}. \tag{4.13}$$

In (4.12) and (4.13), $m \in Z_+$. There is one interesting remark we would like to point out by the end of this section. One notes that in the second case of (4.5), instead of specifying $M^2 = 2$, in general we have

$$\frac{m^2}{R^2} + \frac{1}{4}n^2R^2 = M^2 \tag{4.14}$$

with $mn = 0$. For say $R = \sqrt{2}$, one gets $M^2 = \frac{m^2}{2}(n = 0)$. This means that we have an infinite number of massive SGS at any higher massive level of the spectrum. One can even explicitly write down the vertex operators of these SGS. We conjecture that the w_∞ symmetry of 2D string theory [6,10] can be realized in these SGS [11]. Other moduli points also consist of higher massive SGS in the spectrum.

5 Conclusion

It is hoped that all space-time symmetry of string theory are due to the existence of gauge state in the spectrum. The Heterotic gauge state for the 10D Heterotic string and discrete gauge state for the toy 2D string are such examples. We have introduced soliton gauge state (SGS) for compatified string in this paper, and have related them to the enhenced Kaluza-Klein Kac-Moody gauge symmetry in the theory. In many cases, especially for the massive states, it is easier to study gauge symmetry in the gauge state sector than in the propagating spectrum directly. Since the discrete T-duality symmetry group for bosonic string is the Weyl subgroup of the enhenced gauge group, it can also be considered as implied by the existence of SGS. It is not clear whether other discrete duality symmetry group can be understood in this way. Finally, it would be interesting to consider more complicated compatification, e.g. orbifold and Calabi-Yau compatifications and study the relation between SGS and duality symmetry. works in this direction is in progress.

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